$C(X)$-Algebras and their K-Stability

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Unital C^*-algebra, A

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Amounts to calculating $\pi_k(\mathcal{U}(A))$, $\pi_k(\mathcal{GL}(A))$

Example: The Complex numbers

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- $\mathcal{U}(\mathbb{C}) = S^1$

$$\pi_m(\mathcal{U}(\mathbb{C})) = \begin{cases} \mathbb{Z} & \text{if } m = 1 \\ 0 & \text{if } m \neq 1 \end{cases}$$

- However, for $n > 1$, $\pi_m(\mathcal{U}_n(\mathbb{C}))$ can be very complicated, and typically has torsion. For instance

$$\pi_6(\mathcal{U}_2(\mathbb{C})) = \mathbb{Z}_{12}$$

- By Bott periodicity, if $m > 1$ and $n \geq \frac{m+1}{2}$

$$\pi_m(\mathcal{U}_n(\mathbb{C})) = \begin{cases} 0 & \text{if } m \text{ even} \\ \mathbb{Z} & \text{if } m \text{ odd} \end{cases}$$

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- For the group of unitaries of $n \times n$ matrix algebra over A, denoted by $\mathcal{U}_n(A)$, the homotopy groups $\pi_k(\mathcal{U}_n(A))$ change with respect to the matrix size n, even in simplest case when $A = \mathbb{C}$.

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- In fact, for $M_n(\mathbb{C})$, for $2n \leq k$, $\pi_k(\mathcal{U}_n(\mathbb{C}))$ remain unknown in homotopy theory, as the problem is closely related to the fibration

$$\mathcal{U}_{n-1}(\mathbb{C}) \to \mathcal{U}_n(\mathbb{C}) \to S^{n-1}$$

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- Bott periodicity gives stabilization results for the above groups.

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for all $n \geq 1$ and for all $k \geq 0$?

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  for all $n \geq 1$ and for all $k \geq 0$?
- Are there C^*-algebras for which the nonstable K-groups coincide with the usual K-theory groups?

Unitaries in C^*-Algebras

The Unitary and the Quasi-Unitary Group

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- If B is a unital C^*-algebra, we write $\mathcal{U}(B)$ for the group of unitaries in B.
- For a C^*-algebra A, the map $\widehat{\mathcal{U}}(A) \to \mathcal{U}(A^+)$ given by $u \mapsto 1 - u$, is an isomorphism in case A is unital.

Homology Theory on C^*-Algebras

Definition

For any C^*-algebra A and any $k \geq 0$, we set

$$G_k(A) := \pi_k(\widehat{\mathcal{U}}(A))$$

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Then

- For each k, G_k is a homotopy invariant functor from the category of C^*-algebras to the category of groups.
- Infact, for $k \geq 0$, G_k is a continuous homology theory on the category of C^*-algebras.
- For a unital C^*-algebra A, $G_k(A) \cong \pi_k(\mathcal{U}(A))$.

K-Stable C^*-Algebras

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Let A be a C^*-algebra and $j \geq 2$. Define $\iota_j : M_{j-1}(A) \to M_j(A)$ to be the natural inclusion map

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Definition (Thomsen, 1991)

A C^*-algebra A is said to be *K-stable* if

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Note:

$\mathbb{C}$ is not K-stable. In fact, any finite dimensional C^*-algebra is not K-stable.

Example: The UHF-algebra of type 2^∞

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$A = M_{2^\infty}$ is an inductive limit of

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \dots$$

Where the connecting maps are

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Where the connecting maps are

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The map $\iota_2 : \mathcal{U}(A) \to \mathcal{U}_2(A)$ is then a homotopy equivalence because

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \sim_h \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

in $\mathcal{U}_2(B)$, for any unital C^*-algebra B.

Examples of K-Stable C^*-Algebras

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- [Jiang, 1997] The Jiang-Su algebra $\mathcal{Z}$.

Connection to K-Theory

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Proposition (Thomsen, 1991)

For any C^*-algebra A, $G_k(\mathcal{K} \otimes A)$ is naturally isomorphic to $K_{k+1}(A)$, $k \geq 0$.

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Hence, to say that a C^*-algebra is K-stable, is to say that

$$G_k(A) \cong K_{k+1}(A)$$

Thus, for $A = M_{2^\infty}$, for each $n \in \mathbb{N}$

$$G_k(M_n(A)) \cong K_{k+1}(A) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{2}\right] & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Thus, K-stability gives one of the answer to previously raised questions. However, in the absence of K-stability, we do not, as yet, have any good tools to calculate these homotopy groups.

Rational Homotopy Theory

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- Homotopy theory is the study of spaces with homotopy equivalence. In rational homotopy theory one simplifies these invariants. Instead of $H_n(\cdot)$ and $\pi_n(\cdot)$, we consider $H_n(\cdot;\mathbb{Q})$ and $\pi_n(\cdot)\otimes\mathbb{Q}$.

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- Homotopy theory is the study of spaces with homotopy equivalence. In rational homotopy theory one simplifies these invariants. Instead of $H_n(\cdot)$ and $\pi_n(\cdot)$, we consider $H_n(\cdot;\mathbb{Q})$ and $\pi_n(\cdot)\otimes\mathbb{Q}$.

Theorem (Sullivan)

Let X be a connected H-space of finite type. Then there exists a graded vector space V such that

$$H^*(X;\mathbb{Q}) = \wedge V \text{ and } \pi_*(X)\otimes\mathbb{Q} \cong V^*$$

Furthermore, the construction of V is functorial and the above isomorphism is natural.

Example: Revisiting the Complex Numbers

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For $n \in \mathbb{N}$

$$H^*(\mathcal{U}_n(\mathbb{C}); \mathbb{Q}) \cong \wedge(x_1, x_3, \ldots x_{2n-1})$$

where x_i has degree i. It follows by theorem by Sullivan that

$$\pi_m(\mathcal{U}_n(\mathbb{C})) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } 1 \leq m \leq 2n-1, m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

For the sake of simplicity and ease in computations, along with looking at nonstable K-groups for a given C^*-algebra, for $m \geq 1$, we want to understand/possibly compute the rational nonstable K-groups

$$F_m(A) := \pi_m(\widehat{\mathcal{U}}(A)) \otimes \mathbb{Q}$$

for the C^*-algebra A.

Rationally K-Stable C^*-Algebras

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Definition

A C^*-algebra A is said to be *rationally K-stable* if

$$F_m(\iota_j) : F_m(M_{j-1}(A)) \to F_m(M_j(A))$$

is an isomorphism for all $m \geq 1$ and all $j \geq 2$.

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- The converse need not be true in general.

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- For a rationally K-stable C^*-algebra A, for $m \geq 1$, $F_m(A) \cong K_{m+1}(A) \otimes \mathbb{Q}$.
- For a C^*-algebra A, the property of being K-stable implies being rationally K-stable.
- The converse need not be true in general.
- However, for some classes of C^*-algebras, like the AF-algebras and some A$\mathbb{T}$-algebras, the converse also holds true.

Example

Example

Theorem (Seth, Vaidyanathan, 2021)

There exists a commutative C^*-algebra which is rationally K-stable and not K-stable.

$C(X)$-Algebras

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Definition

Let X be a compact Hausdorff space and A a unital C^*-algebra. A is called a *$C(X)$-algebra* if there is a unital $*$-homomorphism

$$\Delta : C(X) \to Z(A)$$

where $Z(A)$ denotes the centre of A.

In other words, A carries a central $C(X)$-action.

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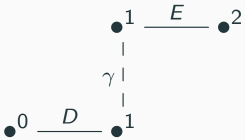
- $A := C(X) \otimes D$ for any unital C^*-algebra D.
- If $\gamma : D \to E$ ia a $*$-homomorphism, then

$$A := \{(f, g) \in C[0,1] \otimes D \oplus C[1,2] \otimes E : \gamma(f(1)) = g(1)\}$$

is a $C[0,2]$-algebra. Pictorially,



Continuous $C(X)$-algebras

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Let A be a $C(X)$-algebra. For $x \in X$, define

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Then I_x is an ideal of $C(X)$, so $I_x \cdot A$ is an ideal of A.

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Definition

The *fiber* of A at x is the quotient

$$A_x := \frac{A}{I_x \cdot A}$$

Continuous $C(X)$-algebras

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For $a \in A$, define $a(x) \in A_x$ to be the image of a in A_x. Define

$$N_a : X \to \mathbb{R} \text{ given by } x \mapsto ||a(x)||$$

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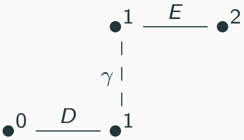
In the example above, if A is given by the picture



then A is a continuous $C[0,2]$-algebra if and only if γ is injective.

Main Result 1

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Theorem (Seth, Vaidyanathan, 2020 (2021))

Let X be a compact metric space of finite covering dimension, and let A be a continuous $C(X)$-algebra. If each fiber of A is K-stable (rationally K-stable), then A is also K-stable (rationally K-stable).

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Theorem (Seth, Vaidyanathan, 2020 (2021))

Let X be a compact metric space of finite covering dimension, and let A be a continuous $C(X)$-algebra. If each fiber of A is K-stable (rationally K-stable), then A is also K-stable (rationally K-stable).

One may think of this as a permanence property for the class of K-stable (rationally K-stable) C^*-algebras.

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- If X is zero-dimensional, then the metrizability condition on X may be dropped.
- If X is not metrizable, we may replace covering dimension with inductive dimension (All notions of dimension coincide for compact metric spaces).
- That X has finite dimension is crucial for the proof, as it works by induction on the dimension.

Extent To Which The Converse Holds

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- Let X be a locally compact, Hausdorff space, and A be a C^*-algebra. If A is rationally K-stable, then so is $C_0(X) \otimes A$. The converse is true if X is a finite CW-complex.

Extent To Which The Converse Holds

- Let X be a locally compact, Hausdorff space, and A be a C^*-algebra. If A is rationally K-stable, then so is $C_0(X) \otimes A$. The converse is true if X is a finite CW-complex.
- Let X be a finite CW-complex, and A be an AF-algebra. Then, $C(X) \otimes A$ is K-stable if and only if A is K-stable.

Application to Crossed Product C^*-Algebras

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Question?

If A is a K-stable or rationally K-stable C^*-algebra, then can we impose conditions on α so that $A \rtimes_\alpha G$ also becomes K-stable or rationally K-stable ?

Rokhlin Dimension

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Definition

Let G be a compact, second countable group, and let A be a separable C^*-algebra. We say that an action $\alpha : G \to \mathrm{Aut}(A)$ has Rokhlin dimension d (with commuting towers) if d is the least integer such that, for any pair of finite sets $F \subset A, K \subset C(G)$, and any $\epsilon > 0$, there exist $(d+1)$ contractive, completely positive maps

$$\psi_0, \psi_1, \ldots, \psi_d : C(G) \to A$$

satisfying the following conditions:

1. For $f_1, f_2 \in K$ such that $f_1 \perp f_2$, $\|\psi_j(f_1)\psi_j(f_2)\| < \epsilon$ for all $0 \leq j \leq d$.

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5. For any $f_1, f_2 \in K$, $\|\psi_j(f_1)\psi_k(f_2) - \psi_k(f_2)\psi_j(f_1)\| < \epsilon$ for all $0 \leq j, k \leq d$.

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We denote the Rokhlin dimension (with commuting towers) of α by $\dim_{Rok}^{c}(\alpha)$. If no such integer exists, we say that α has infinite Rokhlin dimension (with commuting towers), and write $\dim_{Rok}^{c}(\alpha) = +\infty$.

Sequentially Split $*$-Homomorphism

Sequentially Split $*$-Homomorphism

Definition

Let A and B be separable C^*-algebras. A $*$-homomorphism $\varphi : A \to B$ is said to be *sequentially split* if, for every compact set $F \subset A$, and for every $\epsilon > 0$, there exists a $*$-homomorphism $\psi = \psi_{F,\epsilon} : B \to A$ such that

$$\|\psi \circ \phi(a) - a\| < \epsilon$$

for all $a \in F$.

A Permanence Property

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Theorem (Seth, Vaidyanathan, 2021)

Let A and B be separable C^*-algebras, and $\varphi : A \to B$ be a sequentially split $*$-homomorphism. If B is rationally K-stable (K-stable), then so is A.

Local Approximation Theorem

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Theorem (Gardella, Hirshberg, Santiago,2021)

Let G be a compact, second countable group, X be a compact Hausdorff space and A be a separable C^*-algebra. Let $G \curvearrowright X$ be a continuous, free action of G on X, and $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on A. Equip the C^*-algebra $C(X, A)$ with the diagonal action of G, denoted by γ. Then, the crossed product C^*-algebra $C(X, A) \rtimes_\gamma G$ is a continuous $C(X/G)$-algebra, each of whose fibers are isomorphic to $A \otimes \mathcal{K}(L^2(G))$.

Structure Theorem for Crossed Product C^*-Algebras

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Theorem (Gardella, Hirshberg, Santiago, 2021)

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a compact, second countable group on a separable C^*-algebra such that $\dim^c_{Rok}(\alpha) < \infty$. Then, there is exists a compact metric space X and a free action $G \curvearrowright X$ such that the canonical embedding

$$\rho : A \rtimes_\alpha G \to C(X, A) \rtimes_\gamma G$$

is sequentially split. Furthermore, if G finite dimensional, then X may be chosen to be finite dimensional as well.

Main Result 2

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Theorem (Seth, Vaidyanathan, 2021)

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a compact Lie group on a separable C^*-algebra A such that $\dim_{Rok}^{c}(\alpha) < \infty$. If A is K-stable (rationally K-stable), then so is $A \rtimes_\alpha G$.

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- Let X be the (finite dimensional) metric space space obtained from the structure theorem for crossed product C^*-algebra.
- By the local approximation theorem, $C(X, A) \rtimes_\gamma G$ is a continuous $C(X/G)$-algebra, each of whose fibers are isomorphic to $A \otimes \mathcal{K}(L^2(G))$, and are hence K-stable.

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- By main result 1, we conclude that $C(X, A) \rtimes_\gamma G$ is K-stable, and hence $A \rtimes_\alpha G$ is K-stable as a consequence of the permanence property of sequentially split $*$-homomorphism.

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- K-stability and rational K-stability both turn out to be effective tools in understanding the above groups.
- We saw that many interesting simple C^*-algebras are K-stable, hence rationally K-stable. Thus, we wanted to enlarge this class by adding non simple C^*-algebras to it.
- To this end, we showed that the property of K-stability (rational K-stability) passes from the fibers to the ambient algebra provided the underlying space is compact, metrizable and of finite covering dimension

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- We saw that the class of K-stable (rational K-stable) C^*-algebras is closed under the formulation of certain crossed product C^*-algebras.
- In particular we saw, if an action of compact lie group on a separable C^*-algebra has finite Rokhlin dimension (with commuting towers), if A is K-stable (rationally K-stable) then so is the crossed product C^*-algebra.

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Thank you for your time.